Lecture 14.

plan: • Recap §7.2

• Discuss §7.3 properties of $\mathcal{L}\{f\}$.

Recap §7.2

• Laplace transform of f:

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The above might be defined only for some $s \in \mathbb{R}$.

• Linearity of $\mathcal{L}$

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\}(s_0) = c_1 \mathcal{L}\{f_1\}(s_0) + c_2 \mathcal{L}\{f_2\}(s_0)$$

- Important: Table of Laplace transform

$f(t)$	$\mathcal{L}\{f\}(s)$	Region of existence
c (a constant)	$\frac{c}{s}$	$s>0$
t^n, $n>0$ integer	$\frac{n!}{s^{n+1}}$	$s>0$
e^{kt}, $k\in\mathbb{R}$	$\frac{1}{s-k}$	$s>k$
$\sin bt$, $b\in\mathbb{R}$	$\frac{b}{s^2+b^2}$	$s>0$
$\cos bt$, $b\in\mathbb{R}$	$\frac{s}{s^2+b^2}$	$s>0$
$e^{at}\sin(bt)$, $a,b\in\mathbb{R}$	$\frac{b}{(s-a)^2+b^2}$	$s>a$
$e^{at}\cos(bt)$, $a,b\in\mathbb{R}$	$\frac{s-a}{(s-a)^2+b^2}$	$s>a$

§7.3.

Today we will discuss more about Laplace transform.

Thm 1 (Translation):

If $\mathcal{L}\{f\}(s)$ exists for $s > \alpha$, then

β = constant $$\mathcal{L}\{e^{\beta t} f(t)\}(s) = \mathcal{L}\{f\}(s-\beta) \quad (*)$$

for $s > \alpha + \beta$.

Pf: Note $\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt$

Replace "s" by "$s-\beta$" everywhere in the above

$$\Rightarrow \underbrace{\mathcal{L}\{f\}(s-\beta)}_{\text{RHS of } (*)} = \int_0^\infty e^{-(s-\beta)t} f(t)\, dt$$

$$= \int_0^\infty e^{(\beta-s)t} f(t)\, dt$$

$$\underbrace{\mathcal{L}\{e^{\beta t}f(t)\}}_{\text{LHS of }(*)}(s) = \int_0^\infty e^{-st}e^{\beta t}f(t)\,dt$$

$$= \int_0^\infty e^{(\beta-s)t}f(t)\,dt$$

E.g. Given that $\mathcal{L}\{\sin(bt)\}(s) = F(s) = \frac{b}{s^2+b^2}$, $s>0$ b is a given constant

Find $\mathcal{L}\{e^{at}\sin(bt)\}$

A: By the Thm 1, translate by "$-a$"

$$\mathcal{L}\{e^{at}\sin(bt)\} = \mathcal{L}\{\sin(bt)\}(s-a)$$

if $s > \underset{0}{\alpha} + \underset{a}{\beta} = a$

Thus, in the expression of $\mathcal{L}\{\sin(bt)\}$, we replace s by $s-a$. $\Rightarrow$

$$\mathcal{L}\{e^{at}\sin(bt)\}(s) = F(s-a) = \frac{b}{(s-a)^2+b^2}$$

Thm 2 (Derivative).

Let f be continuous on $[0,\infty)$ and assume f' exists. Assume f' is piecewise continuous on $[0,\infty)$. with both of exponential order α. Then for $s>\alpha$,

$\Rightarrow$ $$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0).$$

Pf: Read the book.

E.g: Given

$$\mathcal{L}\{\cos t\}(s) = \frac{s}{s^2+1}, \quad s>0$$

Find $\mathcal{L}\{\sin t\}(s)$.

A: Let $f = \cos t \Rightarrow f' = -\sin t$

They are both Continuous on $[0, \infty)$

Moreover, they are of exponential order 0.

Why? $\begin{cases} |\cos t| \le e^{0 \cdot t} \\ |\sin t| \le e^{0 \cdot t} \end{cases}$ $\alpha = 0$

A function $f(t)$ is said to exponential order α if there exist T, M such that

$|f(t)| \le M e^{\alpha t}$, for all $t \ge T$.

$\Rightarrow$ for $s > 0$

$$\mathcal{L}\{-\sin t\}(s) = \mathcal{L}\{(\cos t)'\}$$

$$-\mathcal{L}\{\sin t\} = s \frac{s}{s^2+1} - \cos 0$$

$$= \frac{s^2}{s^2+1} - 1$$

$$= \frac{-1}{s^2+1}$$

$\Rightarrow$

$$\mathcal{L}\{\sin t\}(s) = \frac{1}{s^2+1}$$

Thm 3. (Integral) The following holds whenever the two Laplace transforms:

$$\mathcal{L}\left\{\int_0^t f(x)dx\right\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(s)$$

E.g. Given $\qquad$ c: constant

$$\mathcal{L}\{c\}(s) \quad \frac{c}{s}, \quad s>0$$

Compute $\mathcal{L}\{t\}(s)$

A: By the Thm 3,

$$\mathcal{L}\{ct\}(s) = \mathcal{L}\left\{\int_0^t c\,dx\right\}(s) = \frac{1}{s}\mathcal{L}\{c\}(s)$$

$$\shortparallel$$

$$c\mathcal{L}\{t\} \qquad = \frac{1}{s}\frac{c}{s} = \frac{c}{s^2}$$

$$\Rightarrow \quad \mathcal{L}\{t\}(s) = \frac{1}{s^2}$$

Thm 4: (Higher-order derivatives)

Assume $f, f', \dots, f^{(n-1)}, f^{(n)}$ are all piecewise continuous on $[0, \infty)$, and are of exponential order α. Then for $s > \alpha$.

$$\mathcal{L}\{f^{(n)}\}(s) = s^n \mathcal{L}\{f\}(s) \underbrace{- s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)}_{-s^{n-1-j} f^{(j)}(0),\ 0 \le j \le n-1}.$$

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0) \leftarrow \text{Thm } 2$$

Ex ① prove the above Thm 4 for $n=2$. by using Thm 2.

By Thm 2,

$$\mathcal{L}\{g'\}(s) = s\mathcal{L}\{g\}(s) - g(0)$$

Let $g = f' \Rightarrow$

$$\mathcal{L}\{f''\}(s) = s\,\mathcal{L}\{f'\}(s) - f'(0)$$

$$= s\left(s\mathcal{L}\{f\}(s) - f(0)\right) - f'(0)$$

$$= s^2 \mathcal{L}\{f\}(s) - s f(0) - f'(0)$$

Hence

☆☆☆ $n=2$

$$\mathcal{L}\{f''\}(s) = s^2 \mathcal{L}\{f\}(s) - s f(0) - f'(0)$$

Remark: To prove Thm 4 in general, one can use proof by induction.

Thm 5: (Multiply of t^n).

Assume $F(s) = \mathcal{L}\{f\}(s)$ for $s > \alpha$.

Then for $s > \alpha$,

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n F}{ds^n}(s)$$

b is a given constant

E.g: Compute $\mathcal{L}\{t\sin(bt)\}(s)$, for $s>0$

A: By the table, we have

$$\mathcal{L}\{\sin bt\}(s) = F(s) = \frac{b}{s^2+b^2}, \quad s>0$$

By Thm 5 with $n=1$, $s>0$

$$\mathcal{L}\{t\sin bt\} = (-1)\frac{dF}{ds}$$

$$= (-1)\left(\frac{b}{s^2+b^2}\right)' = \frac{2bs}{(s^2+b^2)^2}$$

E.g. Compute $\mathcal{L}\{\sin^2 t + e^{3t}t^2\}$.

Hint: $\sin^2 t = \frac{1}{2}(1-\cos(2t))$

A: By the Hint:

$$\mathcal{L}\{\sin^2 t + e^{3t}t^2\}(s)$$

$$= \mathcal{L}\{\tfrac{1}{2}1 - \tfrac{1}{2}\cos(2t) + e^{3t}t^2\}(s)$$

Linearity of $\mathcal{L}$

$$= \underbrace{\tfrac{1}{2}\mathcal{L}\{1\}(s)}_{(\mathrm{I})} - \underbrace{\tfrac{1}{2}\mathcal{L}\{\cos 2t\}(s)}_{(\mathrm{II})} + \underbrace{\mathcal{L}\{e^{3t}t^2\}(s)}_{(\mathrm{III})}$$

By the table,

$$\mathcal{L}\{c\}(s) = \frac{c}{s} \xRightarrow{c=1} \mathcal{L}\{1\}(s) = \frac{1}{s}, \quad s>0$$

$$\mathcal{L}\{\cos 2t\}(s) = \frac{s}{s^2+2^2} = \frac{s}{s^2+4}, \quad s>0$$

$$\mathcal{L}\{f\} = \mathcal{L}\{e^{3t}\}(s) = \underbrace{\frac{1}{s-3}}_{F(s)}, \quad s>3$$

By Thm 5 with $n=2$,

$$\mathcal{L}\{t^2e^{3t}\}(s) = (-1)^2\frac{d^2F}{ds^2}$$

$$= \left(\frac{1}{s-3}\right)''$$

$$= \frac{2}{(s-3)^3} \quad \text{for } s>3$$

Hence

$$\mathcal{L}\{\sin^2 t + e^{3t}t^2\}$$

$$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{(s^2+4)} + \frac{2}{(s-3)^2} \text{ for } s>3$$